

**A new numerical scheme to compute 3D configurations of
quasiequilibrium compact stars in general relativity**
– **Application to synchronously rotating binary star systems** –

Fumihiko Usui

Department of Earth Science and Astronomy,

Graduate School of Arts and Sciences,

University of Tokyo, Komaba, Meguro, Tokyo 153-8902, Japan

Kōji Uryū

SISSA, Via Beirut 2-4, Trieste 34013, Italy

Yoshiharu Eriguchi

Department of Earth Science and Astronomy,

Graduate School of Arts and Sciences,

University of Tokyo, Komaba, Meguro, Tokyo 153-8902, Japan

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Abstract

We have developed a new numerical scheme to obtain quasiequilibrium structures of nonaxisymmetric compact stars such as binary neutron star systems as well as the spacetime around those systems in general relativity. Although, strictly speaking, there are no equilibrium states for binary configurations in general relativity, the timescale of orbital change due to gravitational wave radiation is so long compared with the orbital period that we can assume nonaxisymmetric systems in “quasiequilibrium” states.

Concerning quasiequilibrium states of binary systems in general relativity, several investigations have been already carried out by assuming conformal flatness of the spatial part of the metric. However, the validity of the conformally flat treatment has not been fully analyzed except for axisymmetric configurations. Therefore it is desirable to solve quasiequilibrium states by developing totally different methods from the conformally flat scheme. In this paper we present a new numerical scheme to solve directly the Einstein equations for 3D configurations without assuming conformal flatness, although we make use of the simplified metric for the spacetime. This new formulation is the extension of the scheme which has been successfully applied for structures of axisymmetric rotating compact stars in general relativity. It is based on the integral representation of the Einstein equations by taking the boundary conditions at infinity into account. We have checked our numerical scheme by computing equilibrium sequences of binary polytropic star systems in Newtonian gravity and those of axisymmetric polytropic stars in general relativity. We have applied this numerical code to binary star systems in general relativity and have succeeded in obtaining several equilibrium sequences of synchronously rotating binary polytropes with the polytropic indices $N = 0.0, 0.5$ and 1.0 .

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I. INTRODUCTION

One of the most important matters for theorists of relativistic astrophysics is to construct reliable models for quasiequilibrium configurations of binary neutron star systems. This is because such systems are the most promising sources of gravitational waves which can be observed by the gravitational wave detectors under construction, such as LIGO (USA), VIRGO (France/Italy), GEO (Germany/Great Britain), and TAMA (Japan).

After the discovery of the first binary pulsar PSR1913+16 [1], the orbital change of PSR1913+16 revealed that the existence of gravitational wave did successfully explain the observational results [2]. From more detailed observations of PSR1913+16, much information about neutron stars such as the mass has been obtained and at the same time many other binary pulsar systems have been found [3–7]. Since there are many binary systems composed of neutron stars/black holes, it is estimated that gravitational waves from binary systems will be observed directly once a year in galaxies within 200Mpc [8]. Gravitational wave detectors will become new eyes in the 21st century for uncovering extreme states of the universe.

In order to understand coalescing stages of compact binary systems from observation of gravitational waves, we need to know detailed processes of coalescing or merging phases. This can be done by performing dynamical computations of evolution of binary star systems to a very high accuracy. However, it is still difficult to do highly accurate numerical computations of 3D configurations in general relativity. One reason for that is the very existence of gravitational wave radiation from the systems. There have been many attempts to take this approach and many important results have been obtained (see, e.g., Oohara, Nakamura and Shibata [9], Nakamura [10]), but fully satisfactory results have not been obtained yet.

Recently several groups have begun to attack this problem from a different standpoint. Although binary systems emit gravitational waves, the timescale of its effect on the orbital motion is rather long compared with the orbital period except for the final stage of coalescence because the energy loss rate or the angular momentum loss rate due to gravitational

waves is so small except for such a final phase. The order can be estimated as $O((v/c)^5)$ where v and c are the typical velocity of the system and the velocity of light, respectively. Thus we can assume that the systems are in definite orbits. If the binary systems are synchronously rotating, the systems can be approximately treated to be in equilibrium states by choosing frames which rotate with the same rotational periods as the orbital periods. In other words, we can treat the systems in ‘quasiequilibrium’.

From a different context, in Newtonian gravity, Hachisu and Eriguchi [11–14] obtained first numerically exact results for binary configurations of polytropes including incompressible models. In their papers, secular instability of synchronously rotating binary systems and dynamical instability of asynchronously rotating binary systems were discussed. On the other hand, recent investigations in Newtonian gravity are deeply related to the problem mentioned above. Lai, Rasio and Shapiro [15–18] have studied this problem by employing the ellipsoidal approximation scheme in which configurations are assumed to be exact ellipsoids. More recently Uryu and Eriguchi [19–22] have developed a scheme for irrotational binary star systems and obtained numerically exact stationary sequences. Newly obtained stationary sequences have been analyzed and much information about dynamical instability of evolutionary sequences of polytropic binary star systems has been obtained.

However, Newtonian configurations cannot be directly applied to realistic evolution for binary compact star systems. Thus some authors have studied binary configurations in post-Newtonian regime [23–30]. Post-Newtonian analyses show that the critical angular velocity for instability is increased by 10-15 % compared with that of Newtonian gravity. It is necessary to solve general relativistic models for binary configurations to compute exact critical states against instabilities.

The first numerical results for quasiequilibrium states of binary configurations in full general relativity have been obtained by Wilson, Mathews and Marronetti [31]. In order to treat the problem tractable, they have assumed that the spatial part of the metric is *conformally flat* (the conformally flat condition (CFC), hereafter). One of their results was

surprising: binary neutron stars become unstable and collapse into black holes individually prior to merging (see also Mathews and Wilson [32]). The same problem has been investigated by applying almost the same formulation but by using a different numerical scheme by Baumgarte et al. [33] (also see Cook, Shapiro and Teukolsky [34], Baumgarte et al. [35]). Baumgarte et al. [33] have found that there exist quasiequilibrium sequences of binary systems just prior to merger, i.e. that individual collapse to black holes will not occur (see also [36,37]).

In order to check the validity of the CFC and its accuracy, Cook, Shapiro and Teukolsky [34] have computed equilibrium configurations of axisymmetric rotating polytropes by using two different schemes: 1) the scheme with the CFC and 2) the KEH scheme [38]. Since differences of physical quantities between two schemes are less than 5 %, they have concluded that the scheme with the CFC will give reasonably accurate results even for other situations such as binary configurations. However, its validity is not fully understood [23], for example, it is uncertain whether results from axisymmetric configurations may be applied to nonaxisymmetric situations or not. In this sense, it is desirable to develop different schemes from that with the CFC and to compare results of two or many different schemes for nonaxisymmetric models.

In this paper, we present a new numerical scheme to handle quasiequilibrium states of nonaxisymmetric configurations in general relativity. This scheme is an extension of the numerical scheme for axisymmetric configurations developed by Komatsu, Eriguchi and Hachisu (KEH scheme) [38]. The basic idea used in [38] is to transform the Einstein equations into integral equations by using the Green function for the Laplacian in the flat space. Since we can include the boundary conditions into the Green function, what we need to do is to solve integral equations by some means. In order to adopt the same procedure for nonaxisymmetric models as that for axisymmetric configurations, the crucial step is to make up the Laplacian in the flat space by arranging the Einstein equations. For nonaxisymmetric configurations, however, the Einstein equations for each metric component do not contain Laplacians in the flat space even after proper arrangements. This can be done if we add

appropriate terms to both sides of properly selected component equations of the Einstein equations, although the source terms of those equations contain extra derivatives in addition to the original sources.

As the first step to construct realistic models for binary star systems, we have assumed a simplified form for the metric. The basic equations consist of combinations of dominant components of the Einstein equations for this simplified metric. We have successfully applied the above procedure to make up Laplacians in the flat space and have succeeded in developing a new reliable numerical scheme.

This paper is organized as follows. In section II, we introduce assumptions and basic formulation of the problem. The choice of the simplified metric and the Einstein equations are explained. In section III, we describe the numerical solving method briefly. In section IV, results of our numerical computations are presented. We have tested our new method by comparing our results with (1) those of the axisymmetric configurations of fully relativistic and rapidly rotating polytropes and (2) those of the Newtonian binary systems. In section V, we discuss the validity of the present scheme and the future prospect.

II. ASSUMPTIONS AND BASIC EQUATIONS

A. Assumptions

As mentioned in Introduction, nonaxisymmetric rotating objects cannot be in equilibrium states in the framework of general relativity. It is because gravitational waves carry away the energy as well as the angular momentum from the binary star systems to infinity and so no conserved physical quantities can exist for them. However, for almost all stages except for the last few milliseconds of coalescence of binary systems, the timescale of change of the system due to gravitational waves, τ_{GW} , is much longer than that of the orbital period, τ_{orb} . Specifically, $\tau_{\text{orb}}/\tau_{\text{GW}} \sim (v/c)^5$ and it is less than 1 % even for $v \sim 0.3c$. At the same time, the energy radiated due to gravitational wave emission is only a few percent of the

total energy even when two stars come very close. Therefore, gravitational waves can be neglected for most stages of evolution of binary star systems.

If we neglect the effect of gravitational waves, we can choose a proper rotating frame in which the system is regarded as a stationary one. If we let the angular velocity of this frame seen from the observer at infinity be Ω , the following vector in the inertial frame:

$$\vec{\xi} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} , \quad (1)$$

can be regarded as a Killing vector (see e.g. Bonazzola, Friebe and Gourgoulhon [39,40], Bonazzola, Gourgoulhon and Marck [41]), where t and φ are the time and the azimuthal coordinates, respectively. Thus we assume that nonaxisymmetric systems are in quasiequilibrium states.

Assumptions we will make in this paper are as follows:

- 1) We will treat a binary system which consists of two equal mass stars in a circular orbit, although our code can treat a single nonaxisymmetric body.
- 2) The binary star system is assumed to be in a stationary state in the rotating frame with the angular velocity Ω . In other words, we will neglect gravitational waves from the binary star system.
- 3) Axes of spins of two stars and that of the orbital motion are parallel to each other. A schematic figure of the system is shown in Figure 1. The z -axis is the rotational axis of the orbital motion. The x -axis is connecting two centers of mass of the stars. The intersection of these two axes is defined as the origin of the system. The y -axis is perpendicular to the x - and z -axes. We will call the $x - y$ plane as the equatorial plane.
- 4) Spins of two stars are synchronized to the orbital motion. Each star is rigidly rotating with the angular velocity Ω if seen from a distant place.
- 5) The matter of the star is perfect fluid and a polytropic relation is assumed.

- 6) The matter distribution and the spacetime are assumed to be symmetric about three planes: the equatorial plane, the $y - z$ plane and the $x - z$ plane.

B. Metric and Einstein equations

In addition to the assumptions mentioned above, we further assume the following form for the metric in the spherical coordinates (r, θ, φ) (hereafter we use the units of $c = G = 1$):

$$ds^2 = -e^{2\nu} dt^2 + r^2 \sin^2 \theta e^{2\beta} (d\varphi - \omega dt)^2 + e^{2\alpha} dr^2 + r^2 e^{2\alpha'} d\theta^2, \quad (2)$$

where ν , β , ω , α and α' are the metric coefficients and they are functions of r , θ and φ . This choice of the metric is not the most general one for quasiequilibrium states because we do not take into account tr - and/or $t\theta$ - components of the metric. However, since the purpose of this paper is to show the effectiveness of our new scheme and the $t\varphi$ - component is considered to be the most dominant one among nondiagonal components, we use the above form for the metric in this paper. Furthermore, for simplicity, we assume the following condition:

$$\alpha = \alpha'. \quad (3)$$

This form of the metric becomes exact for stationarily axisymmetric configurations. Although the assumed metric form is incomplete for quasiequilibrium 3D configurations, the essential and technical part of a new numerical scheme for 3D configurations can be shown by choosing this kind of simplified metric and will be extended to more general form of the metric.

As mentioned before, the equation of state for the matter is assumed to be polytropic as follows:

$$p = K\varepsilon^{1+1/N}, \quad (4)$$

where p , K , ε and N are the pressure, a constant, the energy density and the polytropic index, respectively. It should be noted that this polytropic relation is slightly different from

that used in [33]. This equation of state can be rewritten as follows by using the Lane-Emden function λ :

$$p = p_c \lambda^{1+N} , \quad (5)$$

$$\varepsilon = \varepsilon_c \lambda^N , \quad (6)$$

where p_c and ε_c are the maximum pressure and the maximum energy density, respectively.

The energy-momentum tensor, $T^{\zeta\eta}$, for the perfect fluid is written as:

$$T^{\zeta\eta} = (\varepsilon + p)u^\zeta u^\eta + p g^{\zeta\eta} , \quad (7)$$

where u^ζ and $g^{\zeta\eta}$ are the four velocity of the matter and the metric, respectively. Throughout this paper, Greek indices run from 0 to 3. Concerning the four velocity, since an observer in the rotating frame sees that the matter is static, we can obtain the following relation by using the condition $u^\zeta u_\zeta = -1$:

$$u^\zeta = \frac{e^{-\nu}}{\sqrt{1-v^2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad (8)$$

where v is the proper velocity of the matter:

$$v = r(\Omega - \omega) \sin \theta e^{\beta-\nu} . \quad (9)$$

After a lengthy but straightforward calculation of the Ricci tensor from the metric (2) (see Appendix), combining the obtained Einstein equations appropriately (see Appendix) and introducing the following two variables ρ and γ by

$$\rho \equiv \beta - \nu , \quad (10)$$

$$\gamma \equiv \beta + \nu , \quad (11)$$

we can obtain the following equations for the metric functions ρ, γ, α and ω :

$$\Delta(\rho e^{\gamma/2}) = S_\rho(r, \theta, \varphi) , \quad (12)$$

$$\Delta(\gamma e^{\gamma/2}) = S_\gamma(r, \theta, \varphi) , \quad (13)$$

$$\Delta\alpha = S_\alpha(r, \theta, \varphi) , \quad (14)$$

$$(\Delta + \frac{2}{r} \frac{\partial}{\partial r} + \frac{2 \cot \theta}{r^2} \frac{\partial}{\partial \theta})(\omega e^{\rho+\gamma/2}) = S_\omega(r, \theta, \varphi) , \quad (15)$$

where

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} , \quad (16)$$

and

$$\begin{aligned} S_\rho(r, \theta, \varphi) = e^{\gamma/2} & \left[-8\pi e^{2\alpha}(\varepsilon + p) \frac{1+v^2}{1-v^2} - \left(\frac{1}{r} \frac{\partial \gamma}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \gamma}{\partial \theta} \right) \right. \\ & - 2 \frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \gamma}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} \right) \\ & - 2(\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \gamma}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} \right) \\ & - r^2 \sin^2 \theta e^{2\rho} \left(\left(\frac{\partial \omega}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right) \\ & + \frac{\rho}{2} \left\{ 16\pi e^{2\alpha} p - \left(\frac{1}{r} \frac{\partial \gamma}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \gamma}{\partial \theta} \right) - \frac{1}{2} \left(\left(\frac{\partial \gamma}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \gamma}{\partial \theta} \right)^2 \right) \right. \\ & - 2 \frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \nu}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\ & + 2(\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \beta}{\partial \varphi^2} + \left(\frac{\partial \beta}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\ & + 2(\Omega - \omega) e^{2\alpha-2\nu} \left(-\frac{\partial^2 \omega}{\partial \varphi^2} - \frac{\partial \omega}{\partial \varphi} \left(2 \frac{\partial \alpha}{\partial \varphi} + 2 \frac{\partial \rho}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} \right) \right. \\ & \left. \left. + 2e^{2\alpha-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 \right) \right\} \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} (\rho e^{\gamma/2}) , \end{aligned} \quad (17)$$

$$\begin{aligned} S_\gamma(r, \theta, \varphi) = e^{\gamma/2} & \left[16\pi e^{2\alpha} p - \left(\frac{1}{r} \frac{\partial \gamma}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \gamma}{\partial \theta} \right) \right. \\ & + \frac{\gamma}{2} \left\{ 16\pi e^{2\alpha} p - \left(\frac{1}{r} \frac{\partial \gamma}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \gamma}{\partial \theta} \right) - \frac{1}{2} \left(\left(\frac{\partial \gamma}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \gamma}{\partial \theta} \right)^2 \right) \right. \\ & \left. - 2 \frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \nu}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \right. \end{aligned}$$

$$\begin{aligned}
& + 2(\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \beta}{\partial \varphi^2} + \left(\frac{\partial \beta}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\
& + 2(\Omega - \omega) e^{2\alpha-2\nu} \left(-\frac{\partial^2 \omega}{\partial \varphi^2} - \frac{\partial \omega}{\partial \varphi} \left(2\frac{\partial \alpha}{\partial \varphi} + 2\frac{\partial \rho}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} \right) \right. \\
& \quad \left. + 2e^{2\alpha-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 \right\} \\
& - 2\frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \nu}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\
& + 2(\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \beta}{\partial \varphi^2} + \left(\frac{\partial \beta}{\partial \varphi} \right)^2 + \frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + \frac{\partial \rho}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\
& + 2(\Omega - \omega) e^{2\alpha-2\nu} \left(-\frac{\partial^2 \omega}{\partial \varphi^2} - \frac{\partial \omega}{\partial \varphi} \left(2\frac{\partial \alpha}{\partial \varphi} + 2\frac{\partial \rho}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} \right) \right. \\
& \quad \left. + 2e^{2\alpha-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 \right] \\
& + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} (\gamma e^{\gamma/2}) , \tag{18}
\end{aligned}$$

$$\begin{aligned}
S_\alpha(r, \theta, \varphi) = & -4\pi e^{2\alpha} (\varepsilon + p) + \frac{1}{r} \frac{\partial \alpha}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \alpha}{\partial \theta} + \frac{1}{r} \frac{\partial \nu}{\partial r} + \frac{1}{r^2} \cot \theta \frac{\partial \nu}{\partial \theta} \\
& + \frac{\partial \nu}{\partial r} \frac{\partial \beta}{\partial r} + \frac{1}{r^2} \frac{\partial \nu}{\partial \theta} \frac{\partial \beta}{\partial \theta} \\
& + \frac{1}{4} r^2 \sin^2 \theta e^{2\rho} \left(\left(\frac{\partial \omega}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right) \\
& + \frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \nu}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} \right)^2 - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} - \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right) \\
& - (\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \beta}{\partial \varphi^2} + \left(\frac{\partial \beta}{\partial \varphi} \right)^2 - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} - \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right) \\
& - (\Omega - \omega) e^{2\alpha-2\nu} \left(-\frac{\partial^2 \omega}{\partial \varphi^2} - \frac{\partial \omega}{\partial \varphi} \left(2\frac{\partial \rho}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} \right) - e^{2\alpha-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 \right) \\
& + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \alpha}{\partial \varphi^2} , \tag{19}
\end{aligned}$$

$$\begin{aligned}
S_\omega(r, \theta, \varphi) = & e^{\rho+\gamma/2} \left[-16\pi e^{2\alpha} \frac{(\varepsilon + p)(\Omega - \omega)}{1 - v^2} \right. \\
& + \omega \left\{ -8\pi e^{2\alpha} \frac{(1 + v^2)\varepsilon + 2v^2 p}{1 - v^2} \right. \\
& \quad + \frac{1}{r} \left(2\frac{\partial \rho}{\partial r} - \frac{1}{2} \frac{\partial \gamma}{\partial r} \right) + \frac{1}{r^2} \cot \theta \left(2\frac{\partial \rho}{\partial \theta} - \frac{1}{2} \frac{\partial \gamma}{\partial \theta} \right) \\
& \quad \left. + \left(\left(\frac{\partial \rho}{\partial r} \right)^2 - \frac{1}{4} \left(\frac{\partial \gamma}{\partial r} \right)^2 \right) + \frac{1}{4r^2} \left(4\left(\frac{\partial \rho}{\partial \theta} \right)^2 - \left(\frac{\partial \gamma}{\partial \theta} \right)^2 \right) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} \left(\frac{\partial^2 \nu}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} \right)^2 + 3 \left(\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right) - \frac{\partial \alpha}{\partial \varphi} \left(\frac{\partial \rho}{\partial \varphi} + 2 \frac{\partial \nu}{\partial \varphi} \right) - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\
& + (\Omega - \omega)^2 e^{2\alpha-2\nu} \left(\frac{\partial^2 \beta}{\partial \varphi^2} + \left(\frac{\partial \beta}{\partial \varphi} \right)^2 - \left(\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right) \right. \\
& \left. + \left(\frac{\partial \rho}{\partial \varphi} + 2 \frac{\partial \nu}{\partial \varphi} \right) \frac{\partial \alpha}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \frac{\partial \beta}{\partial \varphi} \right) \\
& + (\Omega - \omega) e^{2\alpha-2\nu} \left(-\frac{\partial^2 \omega}{\partial \varphi^2} - \frac{\partial \omega}{\partial \varphi} \left(2 \frac{\partial \alpha}{\partial \varphi} + 2 \frac{\partial \rho}{\partial \varphi} + \frac{\partial \gamma}{\partial \varphi} \right) \right) + e^{2\alpha-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 \\
& - r^2 \sin^2 \theta e^{2\rho} \left(\left(\frac{\partial \omega}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right) \Big\} \\
& - 4 \frac{e^{2\alpha-2\beta}}{r^2 \sin^2 \theta} (\Omega - \omega) \left(\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \alpha}{\partial \varphi} \frac{\partial \gamma}{\partial \varphi} \right) \Big] \\
& + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} (\omega e^{\rho+\gamma/2}) .
\end{aligned} \tag{20}$$

If we multiply $(r \sin \theta \sin \varphi)$ to both sides of Equation (15), we can obtain the following equation:

$$\Delta(r \sin \theta \sin \varphi \cdot \omega e^{\rho+\gamma/2}) = \tilde{S}_\omega(r, \theta, \varphi) , \tag{21}$$

where

$$\tilde{S}_\omega(r, \theta, \varphi) = r \sin \theta \sin \varphi S_\omega(r, \theta, \varphi) + \frac{2 \cos \varphi}{r \sin \theta} \frac{\partial}{\partial \varphi} (\omega e^{\rho+\gamma/2}) . \tag{22}$$

It should be noted that in these equations we make up Laplacians in the flat space by adding terms with derivatives $\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$ to both sides of the equations. The rest of the Einstein equations are not used in this paper. Therefore, our solutions are not numerically exact ones which satisfy all components of the Einstein equations. This is similar to the CFC approach [31,33]. We will discuss the choice of the components of the Einstein equations in a later section.

C. Boundary conditions and integral representation of the Einstein equations

Equations (12) – (14) and (21) can be regarded as Poisson equations for the corresponding quantities if we treat the right hand sides as source terms, although source terms contain unknown quantities.

Under our assumption of quasiequilibrium states or no emission of gravitational waves, we need not consider that there are gravitational waves at infinity. This implies that the metric at infinity must be flat. Therefore, we can transform these equations by using the Green function for the Laplacian in the flat space, $1/|\mathbf{r} - \mathbf{r}'|$, into the integral equations as follows:

$$\rho = -\frac{1}{4\pi} e^{-\gamma/2} \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\varphi' \sin \theta' r'^2 S_\rho(r', \theta', \varphi') \frac{1}{|\mathbf{r} - \mathbf{r}'|} , \quad (23)$$

$$\gamma = -\frac{1}{4\pi} e^{-\gamma/2} \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\varphi' \sin \theta' r'^2 S_\gamma(r', \theta', \varphi') \frac{1}{|\mathbf{r} - \mathbf{r}'|} , \quad (24)$$

$$\alpha = -\frac{1}{4\pi} \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\varphi' \sin \theta' r'^2 S_\alpha(r', \theta', \varphi') \frac{1}{|\mathbf{r} - \mathbf{r}'|} , \quad (25)$$

$$r \sin \theta \sin \varphi \omega = -\frac{1}{4\pi} e^{-(\rho+\gamma/2)} \int_0^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\varphi' \sin \theta' r'^2 \tilde{S}_\omega(r', \theta', \varphi') \frac{1}{|\mathbf{r} - \mathbf{r}'|} . \quad (26)$$

The Green function in these equations is expanded as follows:

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \sum_{n=0}^{\infty} f_n(r, r') [P_n(\cos \theta) P_n(\cos \theta') \\ &\quad + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \theta') \cos m(\varphi - \varphi')] , \end{aligned} \quad (27)$$

where

$$f_n(r, r') = \begin{cases} \frac{1}{r} \left(\frac{r'}{r} \right)^n , & \text{for } r'/r \leq 1 \\ \frac{1}{r'} \left(\frac{r}{r'} \right)^n , & \text{for } r/r' > 1 \end{cases} \quad (28)$$

and P_n and P_n^m are the Legendre polynomials and the associated Legendre functions, respectively. The asymptotic flatness at infinity, $r \rightarrow \infty$,

$$\rho \sim O(1/r) , \quad (29)$$

$$\gamma \sim O(1/r^2) , \quad (30)$$

$$\alpha \sim O(1/r) , \quad (31)$$

$$\omega \sim O(1/r^3) , \quad (32)$$

are satisfied automatically, if the source terms behave properly.

D. Hydrostatic equation

The hydrostatic equation is derived from the conservation law $T^{\zeta\eta}_{;\eta} = 0$ as follows:

$$\nabla p - (\varepsilon + p)\nabla \ln u^t = 0 . \quad (33)$$

By using the polytropic equation (4) and the rigid rotation law, it can be integrated to the following equation:

$$(1 + N) \ln (K \varepsilon^{1/N} + 1) + \nu + \frac{1}{2} \ln(1 - v^2) = C , \quad (34)$$

where C is a constant of integration.

III. METHOD OF SOLUTION

A. Model parameters

Once the equation of state is fixed, we need to specify two parameters to determine one model for a rotating equilibrium configuration: one parameter which represents the strength of gravity and the other for the amount of rotation. For polytropes, we have to specify one more parameter because the constant K is a free parameter. In our formulation, we choose the following three parameters: 1) the maximum energy density, ε_c , 2) the ratio of the maximum pressure to the maximum energy density, κ :

$$\kappa \equiv \frac{p_c}{\varepsilon_c} , \quad (35)$$

and 3) the ratio of the shortest distance r_A (distance from the origin to point A, see Figure 1) to the largest distance r_B (distance from the origin to point B, see Figure 1), q :

$$q \equiv \frac{r_A}{r_B} . \quad (36)$$

The value of the quantity κ represents the strength of gravity because $\kappa \sim \frac{r_g}{R}$, where r_g and R are the Schwarzschild radius and the radius of the star, respectively. This is regarded as

the parameter of compactness. For post-Newtonian models, $\kappa \leq 0.1$ and for typical neutron stars, $\kappa \sim 0.2 - 0.4$.

The quantity q indirectly specifies the rotation rate. It may be possible to choose alternatively the angular velocity or the angular momentum to specify the rotation rate. However, from the numerical computational point of view, it is much better to choose the ratio of two distances instead of choosing the other physical quantities [13].

For polytropes, we can introduce dimensionless physical quantities by using two constants c and G as well as the maximum energy density ε_c . Since the quantity ε_c does not appear in the basic equations, we can treat the problem with two parameters, κ and q , in addition to the polytropic index N . In summary, once we specify these parameters, all we have to do is to solve for the metric coefficients ρ , γ , α and ω , distributions of the energy density ε , the pressure p and the angular velocity of the system Ω .

In actual numerical computations, we further normalize the radial coordinate and the metric functions as follows:

$$r = r_0 \sqrt{f_x} \tilde{r} , \quad (37)$$

and

$$\rho = f_x \tilde{\rho} , \quad (38)$$

$$\gamma = f_x \tilde{\gamma} , \quad (39)$$

$$\alpha = f_x \tilde{\alpha} , \quad (40)$$

$$\omega = \Omega_0 f_x \tilde{\omega} , \quad (41)$$

where

$$r_0 \equiv \frac{c^2}{\sqrt{G\varepsilon_c}} , \quad (42)$$

$$\Omega_0 \equiv \sqrt{4\pi G\varepsilon_c/c^2} , \quad (43)$$

and f_x is a dimensionless normalization constant which is implicitly determined from

$$r_B = \sqrt{f_x} r_0, \text{ or } \tilde{r}_B = 1. \quad (44)$$

By using these normalized quantities, the hydrostatic equation (34) is rewritten as:

$$(1 + N) \ln(\kappa\lambda + 1) + f_x \tilde{\nu} + \frac{1}{2} \ln(1 - v^2) = C, \quad (45)$$

where

$$v = \sqrt{f_x} \tilde{r} (\tilde{\Omega} - f_x \tilde{\omega}) \sin \theta \exp\{f_x (\tilde{\beta} - \tilde{\nu})\}. \quad (46)$$

Here

$$\tilde{\Omega} \equiv \frac{\Omega}{\Omega_0}. \quad (47)$$

B. Solving scheme

Our formulation in this paper is almost the same as that used in Komatsu, Eriguchi and Hachisu [38]. Therefore we can adopt the HSCF method [13] or the KEH method [38] as our solving scheme. The HSCF method is the extended version of the the Self-Consistent Field method developed by Ostriker and Mark [42].

The essence of the HSCF method or the KEH method can be briefly summarized as follows: At the beginning of the computation, we prepare initial guesses for the metric potentials ρ , γ , α and ω , the energy density ε , and the angular velocity Ω as well as the quantity f_x . Substituting them into right hand sides of the integral equations (23)-(26), we obtain new values of ρ , γ , α and ω . At this point, we must solve the angular velocity Ω , a constant of integration C and the scale parameter f_x . These quantities are obtained from the hydrostatic equations (45) at points A, B (in Figure 1) and at point of the maximum density. Using newly obtained ρ , γ , α , ω , Ω , C and f_x , we calculate the new energy density ε from the hydrostatic equation (45).

These newly obtained values of ρ , γ , α , ω , ε , Ω and f_x are used as the trial guesses in the next iteration cycle. We repeat this procedure until the relative differences between new

values and old ones become small enough, i.e., typically less than 10^{-4} . At this point we regard these values as converged ones.

This iteration cycle is carried out by fixing N , κ and q . After we obtain one equilibrium model, we solve other models by changing the axis ratio q . In this way we can obtain a sequence of equilibrium configurations for the same N and κ .

In actual computations, we have used $(r \times \theta \times \varphi) = (128 \times 49 \times 81)$ grid points and we have taken into account the summation in the expansion of the Green function (27) up to 24th term, i.e. $P_{24}^{24}(\cos \theta) \cos 24\varphi$. Although the integral region for integral equations (23) – (26) extends to infinity, we have covered only a finite region $(0 \leq \tilde{r} \leq 2, 0 \leq \theta \leq \pi/2, 0 \leq \varphi \leq \pi/2)$ as was done for axisymmetric configurations in [38]. In Newtonian gravity ($\kappa \sim 10^{-4}$), convergence is reached after 50 or so iterations, while in relativistic models ($\kappa \sim 0.5$), it takes 200 or more iterations until converged solutions are obtained.

C. Physical quantities

Since there exist no exact equilibrium or stationary states, it is impossible to define conserved quantities for nonaxisymmetric configurations in general relativity. However, within our assumption of quasiequilibrium states or no gravitational waves, we can devise to define the approximate gravitational mass and the approximate angular momentum of the system.

As mentioned before, since we neglect gravitational waves at infinity, the metric at infinity can be approximated by the flat spacetime. Thus we assume that the metric function behaves as Equations (29)– (32). Definitions of approximate quantities can be obtained as follows by taking the asymptotic behavior into consideration.

For the approximate angular momentum, from the field equation of ω , we can derive the following equation by arranging terms:

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot r^2 \sin^2 \theta e^{3\beta-\nu} \frac{\partial \omega}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot r^2 \sin^2 \theta e^{3\beta-\nu} \frac{1}{r} \frac{\partial \omega}{\partial \theta} \right) \\ = -16\pi r \sin \theta e^{2\alpha+2\beta} \frac{(\varepsilon + p)v}{1 - v^2} \end{aligned}$$

$$+ 2e^{2\alpha+\beta-\nu}\omega \left[2\frac{\partial^2\alpha}{\partial\varphi^2} + 2\left(\frac{\partial\alpha}{\partial\varphi}\right)^2 - 2\frac{\partial\alpha}{\partial\varphi} \left(\frac{\partial\beta}{\partial\varphi} + \frac{\partial\nu}{\partial\varphi} \right) \right] .$$

By operating $\int r^2 \sin\theta dr d\theta d\varphi$ to this equation, terms on the left hand side of this equation are converted into surface integrals and can be integrated as

$$\begin{aligned} & \int \left\{ \frac{\partial}{\partial r} \left(r^2 \cdot r^2 \sin^2\theta e^{3\beta-\nu} \frac{\partial\omega}{\partial r} \right) dr \right\} \sin\theta d\theta d\varphi + \int \left\{ \frac{\partial}{\partial\theta} \left(\sin\theta \cdot r^2 \sin^2\theta e^{3\beta-\nu} \frac{1}{r} \frac{\partial\omega}{\partial\theta} \right) d\theta \right\} r dr d\varphi \\ &= \int \left[r^4 \sin^2\theta e^{3\beta-\nu} \frac{\partial\omega}{\partial r} \right]_0^\infty \sin\theta d\theta d\varphi + \int \left[\sin^3\theta e^{3\beta-\nu} \frac{\partial\omega}{\partial\theta} \right]_0^\pi r dr d\varphi \\ &= \int \left(r^4 e^{3\beta-\nu} \frac{\partial\omega}{\partial r} \right)_{r=\infty} \sin^3\theta d\theta d\varphi \\ &= -6J \int \sin^3\theta d\theta d\varphi \\ &= -16\pi J , \end{aligned}$$

where we have used the asymptotic behavior $\omega \sim \frac{2J}{r^3}$ and $\frac{\partial\omega}{\partial r} \sim -\frac{6J}{r^4}$ at $r \sim \infty$. By using this relation, we can find

$$\begin{aligned} J &= \int r \sin\theta e^{2\alpha+2\beta} \frac{(\varepsilon+p)v}{1-v^2} r^2 \sin\theta dr d\theta d\varphi \\ &\quad - \frac{1}{4\pi} \int e^{2\alpha+\beta-\nu}\omega \left[\frac{\partial^2\alpha}{\partial\varphi^2} + \frac{\partial\alpha}{\partial\varphi} \left(\frac{\partial\alpha}{\partial\varphi} - 2\frac{\partial\beta}{\partial\varphi} - 2\frac{\partial\nu}{\partial\varphi} \right) \right] r^2 \sin\theta dr d\theta d\varphi . \end{aligned} \quad (48)$$

Here we should note that in the asymptotic flat region, the coefficient of the first term of the expansion of the metric component ω in terms of $1/r$ is interpreted as the twice of the total angular momentum of the system. Therefore, we may define the quantity J as the approximate angular momentum of the nonaxisymmetric configuration.

In the same way, we can define the approximate gravitational mass as follows:

$$\begin{aligned} M &= \int e^{2\alpha+\beta+\nu} \left\{ (\varepsilon+p) \frac{1+v^2}{1-v^2} + 2p \right\} r^2 \sin\theta dr d\theta d\varphi \\ &\quad + \int 2r \sin\theta e^{2\alpha+2\beta} (\varepsilon+p) \frac{v\omega}{1-v^2} r^2 \sin\theta dr d\theta d\varphi , \end{aligned} \quad (49)$$

where we have used the asymptotic behavior $\omega \sim \frac{2J}{r^3}$ and $\frac{\partial\omega}{\partial r} \sim -\frac{6J}{r^4}$ at $r \sim \infty$.

As for the rest mass, it is natural to define as:

$$\begin{aligned} M_0 &= \int \rho_0 u^t \sqrt{-g} \cdot r^2 \sin\theta dr d\theta d\varphi \\ &= \int e^{2\alpha+\beta} \frac{\varepsilon}{(1+p/\varepsilon)^N} \frac{1}{\sqrt{1-v^2}} r^2 \sin\theta dr d\theta d\varphi , \end{aligned} \quad (50)$$

where $\rho_0 = \frac{\varepsilon}{(1 + p/\varepsilon)^N}$ is the baryon mass density.

IV. RESULTS

A. Numerical tests of our new code

We have checked our code by applying it to two cases for which solutions have been obtained by other methods: 1) rotating and *general relativistic axisymmetric sequences* of polytropes and 2) *Newtonian binary sequences* of polytropes.

As mentioned before, our formulation becomes exact for axisymmetric configurations because the metric form (2) is the general one for the axisymmetric space-time and there can be exact stationary states. Komatsu et al. [38] computed general relativistic and axisymmetric rotating polytropes by developing a 2D code. We have computed the same equilibrium sequences by applying our 3D code.

In Figures 2 (a) and (b), we show equilibrium sequences of models starting from $r_p/r_e = 0.9375$ (nearly spherical) to $r_p/r_e = 0.5625$ with $\kappa = 0.0001, 0.25$ and 0.4 for (a) $N = 0$ and (b) $N = 0.5$, respectively, where r_p and r_e are the polar radius and the equatorial radius of axisymmetric configurations, respectively. In these figures, the squared nondimensional angular velocity $\tilde{\Omega}^2$ is plotted against the nondimensional angular momentum \tilde{J} , where

$$\tilde{J} \equiv \frac{J}{(4\pi G c^{2/3} M_0^{10/3} / \varepsilon_c^{1/3})^{1/2}}. \quad (51)$$

Three curves show our 3D computational results and discrete points denote the results of the 2D code. As seen from these figures, our results of 3D code agree well with those of Komatsu et al. [38] to within less than 0.5%.

As for binary sequences, we have carried out computations of Newtonian binary sequences. In our 3D code, Newtonian limit is treated by choosing $\kappa = 0.0001$. We have computed equilibrium sequences for $N = 0.0, 0.5, 1.0$ and 1.5 from $q = 0.0$ (contact phase) to $q = 0.5$. Our results are compared with those of Hachisu [13] in Figure 3.

As seen from this figure, except for $N = 0.0$ models, corresponding models of two different codes agree well each other to within less than 0.5%. For the $N = 0.0$ sequence, some of our models are not in good agreement with those of Newtonian computations. This may be because our 3D code does not treat the surface region of the star carefully for the constant density distributions of $N = 0$ polytropes due to small mesh numbers within the star. In our computations, the star is covered with 64 grid points at most in the r -direction and its number may not be enough to treat drastically changing density distributions. Although it is possible to increase grid points enough to manage a stiff distribution of the density, we did not try it because such a stiff equation of state is not suitable for real neutron stars.

B. Synchronously rotating relativistic binary sequences

In real evolution of compact binary systems, the matter cannot be represented by a simple polytrope and structures of compact stars will change during evolution so that the value of κ must be changing. Such realistic models can be treated by using the realistic equation of state and following quasiequilibrium sequences with constant baryon mass models. However, in this paper, we report only the results of binary sequences with fixed values of N and κ because the main purpose of this paper is to show a new numerical scheme to treat nonaxisymmetric and general relativistic configurations in quasiequilibrium states.

In Figures 4 (a), (b) and (c), we show the results of synchronously rotating binary equilibrium sequences from $q = 0.0$ to $q = 0.5$ with $\kappa = 0.0001, 0.1, 0.3$ and 0.5 , for (a) $N = 0.0$, (b) $N = 0.5$ and (c) $N = 1.0$ polytropes, respectively, (see also Tables in Appendix).

From these figures, we can see that as the strength of gravity is increased, sequences are shifted to the part with a larger value of the dimensionless angular momentum, i.e. at the righter parts of the panels. Similar tendency can be found for Newtonian binary sequences such as Figure 3. In the Newtonian models, models with larger values of N are shifted to the righter part of the panel because of the density concentration to the central

part of the star. As discussed in [38], the effect of general relativity is to increase the mass concentration to the central part of the star. In other words, models with large κ can be regarded as models with higher "effective" polytropic indices. Consequently sequences of larger κ models are located at the part with a larger value of the dimensionless angular momentum in the figure. This situation can be clearly seen in Figure 5. In this figure, equi-density contours are shown for $N = 0.5$ polytropes with $\kappa = 0.0001$ (Newtonian model) and $\kappa = 0.51$ (relativistic model). For the same value of N , the density concentrates to the central region of the stars for relativistic models more than that for Newtonian models.

The other characteristic feature of these figures is that for $N = 0$ sequences there are turning points where the value of the dimensionless angular momentum becomes minimum but that for $N = 0.5$ and $N = 1.0$ sequences there are no turning points except for Newtonian case with $N = 0.5$.

As discussed in Introduction, in the framework of Newtonian gravity, many sequences have been investigated. In particular, binary sequences are known to be connected to axisymmetric ones by way of Jacobi–Dumb-bell shaped sequences [43]. We show the whole relation of several equilibrium sequences of $N = 0$ polytropes, i.e. from Newtonian to relativistic and from axisymmetric to binary sequences as well as the Jacobi–Dumb-bell shaped sequence for Newtonian gravity in Figure 6.

V. DISCUSSION AND CONCLUSIONS

A. Discussion

As discussed before, the metric form (2) is too simplified and may not be suitable for 3D quasiequilibrium configurations. In general, the number of independent metric functions for stationary spacetime can be reduced to six. Thus, the most crucial point may be that we have not included nondiagonal components except for the $t\varphi$ -component. The results obtained in this paper may be affected by taking into account such nondiagonal components

as well as letting $\alpha' \neq \alpha$. It is not easy to estimate the effect of these terms. For stationary configurations, the Einstein equations for the nondiagonal metric components are written as follows (see e.g. [44]):

$$\frac{\partial(\sqrt{\sigma}h^{3/2}w^{ij})}{\partial x^j} = 16\pi h\sqrt{\sigma}\frac{\varepsilon + p}{1 - v^2}v^i, \quad (52)$$

where

$$\sigma \equiv \det \sigma_{ij} , \quad (53)$$

$$\sigma_{ij} \equiv g_{ij} + hg_i g_j , \quad (54)$$

$$g_i \equiv -\frac{g_{ti}}{g_{tt}} , \quad (55)$$

$$h \equiv -g_{tt} , \quad (56)$$

$$w^{ij} \equiv \sigma^{ik}\sigma^{jl}w_{kl} , \quad (57)$$

$$w_{kl} \equiv \frac{\partial g_k}{\partial x^l} - \frac{\partial g_l}{\partial x^k} , \quad (58)$$

$$v^i \equiv \frac{u^i}{\sqrt{h}(1 - g_j u^j)} , \quad (59)$$

$$v^2 \equiv \sigma_{ij}v^i v^j . \quad (60)$$

These equations can be considered to be *linear equations* for g_i with source terms proportional to the velocity v^i if h and σ_{ij} are assumed to be known. This implies that even when matter velocity is along the azimuthal direction, i.e. $v^\varphi \neq 0, v^r = v^\theta = 0$, other nondiagonal components of the metric will emerge. Thus, in general, three nondiagonal components may be of the same order because they are proportional to the value of the 3-velocity. However, since effect of the nondiagonal components is physically interpreted as dragging of the inertial frame to the corresponding direction, it may be natural to consider that the nondiagonal component along the direction of the matter velocity dominates over nondiagonal components along two other directions. Quantitative estimation should be done by extending the present scheme or devising new schemes.

Concerning the form of the metric, we need to discuss the relation between our metric and the CFC scheme. In the investigations which employ the CFC, the following form of the metric has been used [31,33]:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i - \omega^i dt)(dx^j - \omega^j dt) , \quad (61)$$

where

$$\gamma_{ij} = \Psi^4 f_{ij} , \quad (62)$$

and Latin indices run from 1 to 3. Here f_{ij} is the flat space metric and Ψ is the conformal factor, which is independent of t . The spatial part of this metric is written as:

$$g_{ij}dx^i dx^j = \Psi^4(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) . \quad (63)$$

On the other hand, the spatial part of our metric is:

$$g_{ij}dx^i dx^j = e^{2\alpha} dr^2 + e^{2\alpha} r^2 d\theta^2 + e^{2\beta} r^2 \sin^2 \theta d\varphi^2 . \quad (64)$$

Therefore by comparing α and β obtained from our code, we can see how the CFC is satisfied. Cook et al. [34] did the same estimate for axisymmetric models and found that the CFC is well satisfied for rapidly rotating and general relativistic stars.

In Figure 7 distributions of the metric functions in the meridional plane for a selected model are shown. In Figure 8 relative difference of $|\exp(2\alpha) - \exp(2\beta)|/\exp(2\alpha)$ is plotted against the distance from the rotation axis. As seen from this figure, distributions of two metric components α and β are not similar. As far as $\kappa \leq 0.3$, relative differences between α and β are within 5% not only for axisymmetric stars but also for binary star systems. Thus in this range of gravity strength, the CFC may be a good approximation. However, for models with $\kappa \sim 0.5$, deviations are larger than 10%. We note that this conclusion is obtained for our metric (2) which is not the most general one. However, our results show that one should be careful when the CFC is used for extremely relativistic models.

The local flatness near the rotational axis requires that

$$\alpha = \beta , \quad \text{on the rotation axis.} \quad (65)$$

From Figures 7 and 8 the value of α coincides with that of β to within less than 2% on the rotational axis. Thus our numerical code satisfies the local flatness condition to this accuracy.

As seen from Equations (23)-(26), we must cover the whole space from the origin to infinity in the integrations. In actual computations, however, we have covered only a finite region, i.e. the region of integration is limited to $0 \leq \tilde{r} \leq 2$. To check the error due to this truncation of the region, we have solved a few models by setting the size of the region to be twice as large as the former cases : $0 \leq \tilde{r} \leq 4$. The difference of nondimensional physical quantities, such as the mass, the angular momentum and the angular velocity, between these two cases are less than 0.5%. Concerning the approximate angular momentum, since its value depends on the distributions of the metric functions as seen from Equation (48), the obtained approximate angular momentum contains some other error. When we change the integral region from $0 \leq \tilde{r} \leq 2$ to $0 \leq \tilde{r} \leq 1$ in (48), difference is less than 0.2%. Our obtained values can be considered to be accurate enough.

As for the equation of state, we have chosen the relation (4) as our polytropes. Quantitative values in this paper are different from those obtained from other choices of polytrope such as:

$$\varepsilon = \rho_0 + Np, \quad p = K' \rho_0^{1+1/N} . \quad (66)$$

As shown in the previous section, although there exists a turning point along the quasiequilibrium sequence for stiff equations of state, no turning points appear for softer equations of state. The existence of turning points along *properly chosen* equilibrium sequences are deeply related to the occurrence of some kind of instability (see e.g. [19,35]). Therefore, it is important to develop numerical schemes which can compute unstable equilibrium configurations. In this sense, our present scheme is the one which is extended to handle realistic equilibrium sequences.

B. Conclusion

In this paper we have presented a new numerical scheme to handle 3D configurations in quasiequilibrium states in general relativity. By using the new scheme, we have succeeded

in obtaining quasiequilibrium sequences of synchronously rotating binary star systems. We have treated restricted situations about 1) the metric, 2) the equation of state, and 3) the velocity field for the binary systems. The next step of our investigations is to remove some or all of these restrictions by extending the formulation to the more general form of the metric, including realistic equations of state and/or treating irrotational binary star systems. At that stage, we will be able to study turning points or critical configurations by applying the extended formulation.

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APPENDIX A: RICCI TENSOR AND EINSTEIN EQUATIONS IN THE TETRAD SYSTEM

Ricci tensors associated with the metric (2), $R_{(\zeta)(\eta)}$, is explicitly written in the tetrad system as follows:

$$\begin{aligned}
R_{(t)(t)} = & e^{-2\alpha} \left[\frac{\partial^2 \nu}{\partial r^2} + \frac{\partial \nu}{\partial r} \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) + \frac{2}{r} \frac{\partial \nu}{\partial r} \right] \\
& + \frac{e^{-2\alpha}}{r^2} \left[\frac{\partial^2 \nu}{\partial \theta^2} + \frac{\partial \nu}{\partial \theta} \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \right) + \cot \theta \frac{\partial \nu}{\partial \theta} \right] \\
& + \frac{e^{-2\beta}}{r^2 \sin^2 \theta} \left[\frac{\partial^2 \nu}{\partial \varphi^2} + \frac{\partial \nu}{\partial \varphi} \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) + 2 \frac{\partial \nu}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} \right] \\
& - \frac{1}{2} e^{2\beta-2\nu-2\alpha} r^2 \sin^2 \theta \left[\left(\frac{\partial \omega}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right] \\
& - e^{-2\nu} (\omega - \Omega)^2 \left[2 \frac{\partial^2 \alpha}{\partial \varphi^2} + \frac{\partial^2 \beta}{\partial \varphi^2} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + \frac{\partial \beta}{\partial \varphi} \left(\frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \right) - 2 \frac{\partial \nu}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} \right] \\
& - e^{-2\nu} (\omega - \Omega) \left[\frac{\partial^2 \omega}{\partial \varphi^2} + 2 \frac{\partial \omega}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} + \frac{\partial \omega}{\partial \varphi} \left(3 \frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \right) \right] \\
& - e^{-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2, \tag{A1}
\end{aligned}$$

$$\begin{aligned}
R_{(t)(r)} = & -e^{-\alpha-\nu} \left[(\omega - \Omega) \left(\frac{\partial^2 \alpha}{\partial r \partial \varphi} + \frac{\partial^2 \beta}{\partial r \partial \varphi} \right) + \frac{1}{2} \frac{\partial^2 \omega}{\partial r \partial \varphi} + \frac{1}{2} \frac{\partial \omega}{\partial r} \left(\frac{\partial \beta}{\partial \varphi} + \frac{\partial \nu}{\partial \varphi} \right) \right. \\
& + (\omega - \Omega) \left(\frac{\partial \beta}{\partial r} - \frac{\partial \nu}{\partial r} \right) \frac{\partial \beta}{\partial \varphi} - (\omega - \Omega) \frac{\partial \alpha}{\partial \varphi} \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) \\
& \left. + \left(\frac{\partial \beta}{\partial r} - \frac{\partial \nu}{\partial r} \right) \frac{\partial \omega}{\partial \varphi} + \frac{(\omega - \Omega)}{r} \left(\frac{\partial \beta}{\partial \varphi} - \frac{\partial \alpha}{\partial \varphi} \right) + \frac{1}{r} \frac{\partial \omega}{\partial \varphi} \right], \quad (A2)
\end{aligned}$$

$$\begin{aligned}
R_{(t)(\theta)} = & -\frac{e^{-\alpha-\nu}}{r} \left[(\omega - \Omega) \left(\frac{\partial^2 \alpha}{\partial \theta \partial \varphi} + \frac{\partial^2 \beta}{\partial \theta \partial \varphi} \right) + \frac{1}{2} \frac{\partial^2 \omega}{\partial \theta \partial \varphi} + \frac{1}{2} \frac{\partial \omega}{\partial \theta} \left(\frac{\partial \beta}{\partial \varphi} + \frac{\partial \nu}{\partial \varphi} \right) \right. \\
& + (\omega - \Omega) \left(\frac{\partial \beta}{\partial \theta} - \frac{\partial \nu}{\partial \theta} \right) \frac{\partial \beta}{\partial \varphi} - (\omega - \Omega) \frac{\partial \alpha}{\partial \varphi} \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \right) \\
& \left. + \left(\frac{\partial \beta}{\partial \theta} - \frac{\partial \nu}{\partial \theta} \right) \frac{\partial \omega}{\partial \varphi} + \cot \theta (\omega - \Omega) \left(\frac{\partial \alpha}{\partial \varphi} + \frac{\partial \beta}{\partial \varphi} \right) + \cot \theta \frac{\partial \omega}{\partial \varphi} \right], \quad (A3)
\end{aligned}$$

$$\begin{aligned}
R_{(t)(\varphi)} = & \frac{1}{2} e^{\beta-\nu} r \sin \theta \left[e^{-2\alpha} \frac{\partial^2 \omega}{\partial r^2} + e^{-2\alpha} \left(3 \frac{\partial \beta}{\partial r} + \frac{4}{r} - \frac{\partial \nu}{\partial r} \right) \frac{\partial \omega}{\partial r} \right. \\
& + \frac{e^{-2\alpha}}{r^2} \left\{ \frac{\partial^2 \omega}{\partial \theta^2} + \left(3 \frac{\partial \beta}{\partial \theta} + 3 \cot \theta - \frac{\partial \nu}{\partial \theta} \right) \frac{\partial \omega}{\partial \theta} \right\} \Bigg] \\
& + \frac{e^{-\beta-\nu}}{r \sin \theta} (\omega - \Omega) \left[-2 \frac{\partial^2 \alpha}{\partial \varphi^2} - 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + 2 \left(\frac{\partial \beta}{\partial \varphi} + \frac{\partial \nu}{\partial \varphi} \right) \frac{\partial \alpha}{\partial \varphi} \right], \quad (A4)
\end{aligned}$$

$$\begin{aligned}
R_{(r)(r)} = & -e^{-2\alpha} \left[\frac{\partial^2 \nu}{\partial r^2} + \frac{\partial^2 \beta}{\partial r^2} + \frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \nu}{\partial r} \right)^2 + \left(\frac{\partial \beta}{\partial r} \right)^2 - \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) \frac{\partial \alpha}{\partial r} + \frac{2}{r} \frac{\partial \beta}{\partial r} \right] \\
& - \frac{e^{-2\alpha}}{r^2} \left[\frac{\partial^2 \alpha}{\partial \theta^2} + \cot \theta \frac{\partial \alpha}{\partial \theta} + \frac{\partial \alpha}{\partial \theta} \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \right) \right] \\
& - \frac{e^{-2\beta}}{r^2 \sin^2 \theta} \left[\frac{\partial^2 \alpha}{\partial \varphi^2} + \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) \frac{\partial \alpha}{\partial \varphi} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right] \\
& + e^{-2\nu} (\omega - \Omega)^2 \left[\frac{\partial^2 \alpha}{\partial \varphi^2} - \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) \frac{\partial \alpha}{\partial \varphi} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 \right] \\
& + 2e^{-2\nu} (\omega - \Omega) \frac{\partial \alpha}{\partial \varphi} \frac{\partial \omega}{\partial \varphi} + \frac{1}{2} e^{2\beta-2\nu-2\alpha} r^2 \sin^2 \theta \left(\frac{\partial \omega}{\partial r} \right)^2, \quad (A5)
\end{aligned}$$

$$R_{(r)(\theta)} = -\frac{e^{-2\alpha}}{r} \left[\frac{\partial^2 \nu}{\partial r \partial \theta} + \frac{\partial^2 \beta}{\partial r \partial \theta} + \frac{\partial \nu}{\partial r} \frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial r} \frac{\partial \beta}{\partial \theta} - \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) \frac{\partial \alpha}{\partial \theta} - \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \right) \frac{\partial \alpha}{\partial r} \right. \\ \left. + \cot \theta \frac{\partial \beta}{\partial r} - \cot \theta \frac{\partial \alpha}{\partial r} - \frac{1}{r} \frac{\partial \nu}{\partial \theta} - \frac{1}{r} \frac{\partial \alpha}{\partial \theta} \right], \quad (\text{A6})$$

$$R_{(r)(\varphi)} = -\frac{e^{-\alpha-\beta}}{r \sin \theta} \left[\frac{\partial^2 \nu}{\partial r \partial \varphi} + \frac{\partial^2 \alpha}{\partial r \partial \varphi} + \left(\frac{\partial \nu}{\partial r} - \frac{\partial \beta}{\partial r} \right) \frac{\partial \nu}{\partial \varphi} - \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) \frac{\partial \alpha}{\partial \varphi} - \frac{1}{r} \left(\frac{\partial \alpha}{\partial \varphi} + \frac{\partial \nu}{\partial \varphi} \right) \right. \\ \left. - \frac{1}{2} e^{2\beta-2\nu} r^2 \sin^2 \theta \left\{ (\omega - \Omega) \frac{\partial^2 \omega}{\partial r \partial \varphi} + (\omega - \Omega) \frac{\partial \omega}{\partial r} \left(3 \frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \right) + 2 \frac{\partial \omega}{\partial r} \frac{\partial \omega}{\partial \varphi} \right\} \right], \quad (\text{A7})$$

$$R_{(\theta)(\theta)} = -e^{-2\alpha} \left[\frac{\partial^2 \alpha}{\partial r^2} + \frac{2}{r} \frac{\partial \alpha}{\partial r} + \left(\frac{\partial \alpha}{\partial r} + \frac{1}{r} \right) \left(\frac{\partial \nu}{\partial r} + \frac{\partial \beta}{\partial r} \right) \right. \\ \left. - \frac{e^{-2\alpha}}{r^2} \left[\frac{\partial^2 \alpha}{\partial \theta^2} + \frac{\partial^2 \beta}{\partial \theta^2} + \frac{\partial^2 \nu}{\partial \theta^2} + \left(\frac{\partial \nu}{\partial \theta} \right)^2 + \left(\frac{\partial \beta}{\partial \theta} \right)^2 \right. \right. \\ \left. \left. - \frac{\partial \alpha}{\partial \theta} \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \right) - \cot \theta \frac{\partial \alpha}{\partial \theta} + 2 \cot \theta \frac{\partial \beta}{\partial \theta} \right] \right. \\ \left. - \frac{e^{-2\beta}}{r^2 \sin^2 \theta} \left[\frac{\partial^2 \alpha}{\partial \varphi^2} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 + \frac{\partial \alpha}{\partial \varphi} \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) \right] \right. \\ \left. + e^{-2\nu} (\omega - \Omega)^2 \left[\frac{\partial^2 \alpha}{\partial \varphi^2} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - \frac{\partial \alpha}{\partial \varphi} \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) \right] \right. \\ \left. + 2e^{-2\nu} (\omega - \Omega) \frac{\partial \alpha}{\partial \varphi} \frac{\partial \omega}{\partial \varphi} + \frac{1}{2} e^{2\beta-2\nu-2\alpha} \sin^2 \theta \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right], \quad (\text{A8})$$

$$R_{(\theta)(\varphi)} = -\frac{e^{-\beta-\alpha}}{r^2 \sin^2 \theta} \left[\frac{\partial^2 \nu}{\partial \theta \partial \varphi} + \frac{\partial^2 \alpha}{\partial \theta \partial \varphi} + \left(\frac{\partial \nu}{\partial \theta} - \frac{\partial \beta}{\partial \theta} - \cot \theta \right) \frac{\partial \nu}{\partial \varphi} - \left(\frac{\partial \nu}{\partial \theta} + \frac{\partial \beta}{\partial \theta} + \cot \theta \right) \frac{\partial \alpha}{\partial \varphi} \right. \\ \left. - \frac{1}{2} e^{2\beta-2\nu} r^2 \sin^2 \theta \left\{ (\omega - \Omega) \frac{\partial^2 \omega}{\partial \theta \partial \varphi} + (\omega - \Omega) \frac{\partial \omega}{\partial \theta} \left(3 \frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \right) + 2 \frac{\partial \omega}{\partial \theta} \frac{\partial \omega}{\partial \varphi} \right\} \right], \quad (\text{A9})$$

$$\begin{aligned}
R_{(\varphi)(\varphi)} = & -e^{-2\alpha} \left[\frac{\partial^2 \beta}{\partial r^2} + \frac{3}{r} \frac{\partial \beta}{\partial r} + \frac{\partial \beta}{\partial r} \left(\frac{\partial \beta}{\partial r} + \frac{\partial \alpha}{\partial r} \right) + \frac{1}{r} \frac{\partial \nu}{\partial r} + \frac{1}{r^2} \right] \\
& - \frac{e^{-2\alpha}}{r^2} \left[\frac{\partial^2 \beta}{\partial \theta^2} + 2 \cot \theta \frac{\partial \beta}{\partial \theta} + \frac{\partial \beta}{\partial \theta} \left(\frac{\partial \beta}{\partial \theta} + \frac{\partial \nu}{\partial \theta} \right) + \cot \theta \frac{\partial \nu}{\partial \theta} - 1 \right] \\
& - \frac{e^{-2\beta}}{r^2 \sin^2 \theta} \left[2 \frac{\partial^2 \alpha}{\partial \varphi^2} + \frac{\partial^2 \nu}{\partial \varphi^2} + 2 \left(\frac{\partial \alpha}{\partial \varphi} \right)^2 - 2 \frac{\partial \beta}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} + \frac{\partial \nu}{\partial \varphi} \left(\frac{\partial \nu}{\partial \varphi} - \frac{\partial \beta}{\partial \varphi} \right) \right] \\
& + e^{-2\nu} (\omega - \Omega)^2 \left[\frac{\partial^2 \beta}{\partial \varphi^2} + \frac{\partial \beta}{\partial \varphi} \left(\frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} \right) + 2 \frac{\partial \beta}{\partial \varphi} \frac{\partial \alpha}{\partial \varphi} \right] \\
& + e^{-2\nu} (\omega - \Omega) \left[\frac{\partial^2 \omega}{\partial \varphi^2} + \frac{\partial \omega}{\partial \varphi} \left(3 \frac{\partial \beta}{\partial \varphi} - \frac{\partial \nu}{\partial \varphi} + 2 \frac{\partial \alpha}{\partial \varphi} \right) \right] \\
& + e^{-2\nu} \left(\frac{\partial \omega}{\partial \varphi} \right)^2 - \frac{1}{2} e^{2\beta-2\nu-2\alpha} r^2 \sin^2 \theta \left\{ \left(\frac{\partial \omega}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \omega}{\partial \theta} \right)^2 \right\} , \tag{A10}
\end{aligned}$$

where we have used the orthonormal tetrad defined as:

$$\lambda_{(\eta)}^\xi = \begin{pmatrix} e^{-\nu}, & 0, & 0, & (\omega - \Omega)e^{-\nu} \\ 0, & e^{-\alpha}, & 0, & 0 \\ 0, & 0, & \frac{e^{-\alpha}}{r}, & 0 \\ 0, & 0, & 0, & \frac{e^{-\beta}}{r \sin \theta} \end{pmatrix} , \tag{A11}$$

and

$$\lambda_{(\eta)\xi} = \begin{pmatrix} -e^{-\nu}, & 0, & 0, & 0 \\ 0, & e^\alpha, & 0, & 0 \\ 0, & 0, & r e^\alpha, & 0 \\ -r \sin \theta e^\beta (\omega - \Omega), & 0, & 0, & r \sin \theta e^\beta \end{pmatrix} . \tag{A12}$$

Any tensor, $h_{\zeta\eta}$, is transformed to that in the tetrad system as follows:

$$h_{(\alpha)(\beta)} = h_{\zeta\eta} \lambda_{(\alpha)}^\zeta \lambda_{(\beta)}^\eta . \tag{A13}$$

The energy-momentum tensor for the quasiequilibrium configurations in this paper are:

$$T_{(t)(t)} = \frac{\varepsilon + pv^2}{1 - v^2} , \quad (\text{A14})$$

$$T_{(r)(r)} = T_{(\theta)(\theta)} = p , \quad (\text{A15})$$

$$T_{(\varphi)(\varphi)} = \frac{p + \varepsilon v^2}{1 - v^2} , \quad (\text{A16})$$

$$T_{(t)(\varphi)} = T_{(\varphi)(t)} = -\frac{(\varepsilon + p)v}{1 - v^2} , \quad (\text{A17})$$

$$\text{others} = 0 . \quad (\text{A18})$$

Einstein equation is formally written as:

$$R_{(\zeta)(\eta)} = 8\pi(T_{(\zeta)(\eta)} - \frac{1}{2}g_{(\zeta)(\eta)}T) , \quad (\text{A19})$$

where $T = T_{(\zeta)}^{(\zeta)}$. We have used the following combinations to get elliptic type equations (12)–(14) and (21).

$$R_{(t)(t)} + R_{(\varphi)(\varphi)} = 8\pi(\varepsilon + p)\frac{1 + v^2}{1 - v^2} , \quad (\text{A20})$$

$$R_{(t)(t)} - R_{(\varphi)(\varphi)} = 16\pi p , \quad (\text{A21})$$

$$R_{(t)(t)} + R_{(r)(r)} + R_{(\theta)(\theta)} - R_{(\varphi)(\varphi)} = 8\pi(\varepsilon + p) , \quad (\text{A22})$$

$$R_{(t)(\varphi)} = 8\pi(\varepsilon + p)\frac{v}{1 - v^2} , \quad (\text{A23})$$

APPENDIX B: DETAILED NUMERICAL RESULTS FOR BINARY SEQUENCES

Numerical results of binary sequences are shown in Tables I–XIII.

In these Tables, \tilde{r}_A , \tilde{J} , \tilde{M} , \tilde{M}_0 , $\tilde{\Omega}^2$ and f_x are the dimensionless distance to point A from the origin, the dimensionless total angular momentum, the dimensionless total gravitational mass, the dimensionless total rest mass, the dimensionless angular velocity and the scale parameter, respectively. Here, masses are normalized as:

$$\tilde{M} \equiv \frac{M}{f_x^{3/2} c^4 / (G^{3/2} \varepsilon_c^{1/2})} , \quad (\text{B1})$$

$$\tilde{M}_0 \equiv \frac{M_0}{f_x^{3/2} c^4 / (G^{3/2} \varepsilon_c^{1/2})} . \quad (\text{B2})$$

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FIGURES

FIG. 1. Schematic view of the binary system. The x -axis is set along the line joining the two centers of mass of the two stars and the origin is set on the middle point. The z -axis is along the rotation axis. Point A, B are set at the intersections of the surface of the star and the x -axis. The inner intersection is point A and the outer one is point B.

FIG. 2. Dimensionless squared angular velocity $\Omega^2/(4\pi G\varepsilon_c/c^2)$ is plotted against dimensionless angular momentum $J/(4\pi Gc^{2/3}M_0^{10/3}/\varepsilon_c^{1/3})^{1/2}$ for the sequences of the axisymmetric stars from $r_p/r_e = 0.9375$ (nearly spherical) to $r_p/r_e = 0.5625$ with polytropic indices (a) $N = 0.0$ and (b) $N = 0.5$, where r_p and r_e are the polar radius and the equatorial radius of axisymmetric configurations. The solid line ($\kappa = 0.0001$), the dotted line ($\kappa = 0.25$) and the dashed line ($\kappa = 0.4$) are results obtained by using our 3D code. Crosses ($\kappa = 0.0001$), open circles ($\kappa = 0.25$) and filled circles ($\kappa = 0.4$) are taken from Komatsu et al. [38].

FIG. 3. Same as Figure 2 but for the sequences of Newtonian binary systems from $q = 0.0$ to $q = 0.5$. The solid line ($N = 0.0$), the dotted line ($N = 0.5$), the short-dashed line ($N = 1.0$) and the long-dashed line ($N = 1.5$) are our numerical results. Crosses ($N = 0.0$), filled circles ($N = 0.5$), open circles ($N = 1.0$) and filled rectangles ($N = 1.5$) are taken from Hachisu [13].

FIG. 4. Same as Figure 2 but for the sequences of relativistic binary systems from $q = 0.0$ to $q = 0.5$ with polytropic indices (a) $N = 0.0$, (b) $N = 0.5$ and (c) $N = 1.0$. On each panel, $\kappa = 0.0001$ (Newtonian), 0.1, 0.3 and 0.5 are shown by solid lines. Crosses denote the Newtonian results taken from Hachisu [13].

FIG. 5. Contours of the energy density on the $x - z$ plane. The units of the distance is r_B . (a) The contact phase ($q = 0.0$) and (b) the distant phase ($q = 0.5$) are shown for $N = 0.5$ polytrope. On each panel, contours for $\kappa = 0.0001$ (Newtonian) and $\kappa = 0.5$ (relativistic) are shown.

FIG. 6. Same as Figure 2 but for $N = 0.0$ polytropes with different κ and different topology, i.e. axisymmetric sequences, Jacobi–Dumb-bell shaped sequence and binary sequences. Solid, dotted and dashed curves denote sequences with $\kappa = 0.0001$ (Newtonian), 0.25 and 0.4, respectively. The dash-dotted curve denotes the Jacobi–Dumb-bell shaped sequences taken from Hachisu [13].

FIG. 7. (a) Distribution of the metric function $\exp(2\nu)$ on the equatorial plane is plotted against the distance from the rotational axis. Model parameters are $N = 0.5$, $\kappa = 0.5$ and $q = 0.0$. Different curves correspond to distributions on $\varphi = 0, \pi/12, 2\pi/12, 3\pi/12, 4\pi/12, 5\pi/12, 6\pi/12$. (b) Distribution of the metric function $\exp(2\beta)$. (c) Distribution of the metric function $\exp(2\alpha)$. (d) Distribution of the metric function ω/Ω .

FIG. 8. Distribution of $|\exp(2\alpha) - \exp(2\beta)|/\exp(2\alpha)$ on the equatorial plane is plotted against the distance from the rotational axis. Model parameters are $N = 0.5$, $\kappa = 0.5$ and $q = 0.0$. Different curves correspond to distributions on $\varphi = 0, 3\pi/12, 6\pi/12$.

TABLES

TABLE I. $N = 0.0$, $\kappa = 0.0001$ (Newtonian)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.126E-01	3.246E-06	3.246E-06	2.647E-02	3.984E-04
3.125E-02	1.121E-01	3.233E-06	3.234E-06	2.644E-02	3.960E-04
6.250E-02	1.109E-01	3.187E-06	3.187E-06	2.586E-02	3.894E-04
9.375E-02	1.097E-01	3.137E-06	3.138E-06	2.465E-02	3.836E-04
1.250E-01	1.082E-01	3.078E-06	3.078E-06	2.275E-02	3.797E-04
1.562E-01	1.078E-01	3.032E-06	3.033E-06	2.077E-02	3.830E-04
1.875E-01	1.074E-01	2.991E-06	2.992E-06	1.849E-02	3.893E-04
2.188E-01	1.078E-01	2.965E-06	2.965E-06	1.631E-02	4.013E-04
2.500E-01	1.087E-01	2.938E-06	2.938E-06	1.423E-02	4.176E-04
2.812E-01	1.096E-01	2.905E-06	2.906E-06	1.213E-02	4.379E-04
3.125E-01	1.112E-01	2.874E-06	2.874E-06	1.033E-02	4.620E-04
3.438E-01	1.140E-01	2.825E-06	2.826E-06	8.790E-03	4.914E-04
3.750E-01	1.150E-01	2.780E-06	2.780E-06	7.110E-03	5.261E-04
4.062E-01	1.182E-01	2.752E-06	2.753E-06	5.880E-03	5.713E-04
4.375E-01	1.212E-01	2.750E-06	2.750E-06	4.808E-03	6.262E-04
4.688E-01	1.201E-01	2.780E-06	2.781E-06	3.535E-03	7.014E-04
5.000E-01	1.295E-01	2.820E-06	2.821E-06	3.089E-03	7.916E-04

TABLE II. $N = 0.5$, $\kappa = 0.0001$ (Newtonian)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.145E-01	4.342E-06	4.342E-06	1.668E-02	6.474E-04
3.125E-02	1.142E-01	4.330E-06	4.330E-06	1.647E-02	6.464E-04
6.250E-02	1.137E-01	4.297E-06	4.298E-06	1.582E-02	6.455E-04
9.375E-02	1.132E-01	4.256E-06	4.257E-06	1.480E-02	6.477E-04
1.250E-01	1.130E-01	4.209E-06	4.210E-06	1.353E-02	6.549E-04
1.562E-01	1.131E-01	4.168E-06	4.169E-06	1.213E-02	6.684E-04
1.875E-01	1.136E-01	4.133E-06	4.134E-06	1.070E-02	6.891E-04
2.188E-01	1.145E-01	4.101E-06	4.102E-06	9.309E-03	7.168E-04
2.500E-01	1.158E-01	4.071E-06	4.072E-06	7.998E-03	7.523E-04
2.812E-01	1.175E-01	4.041E-06	4.042E-06	6.800E-03	7.958E-04
3.125E-01	1.195E-01	4.009E-06	4.010E-06	5.711E-03	8.484E-04
3.438E-01	1.219E-01	3.980E-06	3.980E-06	4.746E-03	9.116E-04
3.750E-01	1.246E-01	3.957E-06	3.957E-06	3.900E-03	9.877E-04
4.062E-01	1.278E-01	3.948E-06	3.949E-06	3.188E-03	1.079E-03
4.375E-01	1.316E-01	3.955E-06	3.955E-06	2.592E-03	1.190E-03
4.688E-01	1.349E-01	3.968E-06	3.969E-06	2.052E-03	1.325E-03
5.000E-01	1.408E-01	3.982E-06	3.982E-06	1.666E-03	1.486E-03

TABLE III. $N = 1.0$, $\kappa = 0.0001$ (Newtonian)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.182E-01	5.336E-06	5.337E-06	9.957E-03	1.028E-03
3.125E-02	1.182E-01	5.327E-06	5.328E-06	9.802E-03	1.030E-03
6.250E-02	1.183E-01	5.303E-06	5.304E-06	9.359E-03	1.037E-03
9.375E-02	1.184E-01	5.274E-06	5.274E-06	8.702E-03	1.051E-03
1.250E-01	1.189E-01	5.238E-06	5.239E-06	7.903E-03	1.073E-03
1.562E-01	1.197E-01	5.206E-06	5.207E-06	7.044E-03	1.105E-03
1.875E-01	1.208E-01	5.178E-06	5.178E-06	6.181E-03	1.147E-03
2.188E-01	1.223E-01	5.148E-06	5.149E-06	5.345E-03	1.201E-03
2.500E-01	1.241E-01	5.121E-06	5.122E-06	4.568E-03	1.268E-03
2.812E-01	1.264E-01	5.096E-06	5.097E-06	3.861E-03	1.349E-03
3.125E-01	1.289E-01	5.073E-06	5.074E-06	3.231E-03	1.446E-03
3.438E-01	1.319E-01	5.055E-06	5.056E-06	2.679E-03	1.561E-03
3.750E-01	1.352E-01	5.043E-06	5.043E-06	2.201E-03	1.697E-03
4.062E-01	1.390E-01	5.037E-06	5.038E-06	1.796E-03	1.859E-03
4.375E-01	1.431E-01	5.035E-06	5.036E-06	1.451E-03	2.052E-03
4.688E-01	1.477E-01	5.031E-06	5.032E-06	1.158E-03	2.281E-03
5.000E-01	1.532E-01	5.021E-06	5.022E-06	9.183E-04	2.558E-03

TABLE IV. $N = 1.5$, $\kappa = 0.0001$ (Newtonian)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.246E-01	6.282E-06	6.282E-06	5.596E-03	1.662E-03
3.125E-02	1.247E-01	6.278E-06	6.278E-06	5.509E-03	1.667E-03
6.250E-02	1.251E-01	6.260E-06	6.261E-06	5.252E-03	1.684E-03
9.375E-02	1.258E-01	6.240E-06	6.240E-06	4.878E-03	1.714E-03
1.250E-01	1.268E-01	6.214E-06	6.215E-06	4.422E-03	1.758E-03
1.562E-01	1.281E-01	6.191E-06	6.192E-06	3.936E-03	1.818E-03
1.875E-01	1.297E-01	6.170E-06	6.170E-06	3.448E-03	1.896E-03
2.188E-01	1.318E-01	6.146E-06	6.146E-06	2.975E-03	1.993E-03
2.500E-01	1.342E-01	6.124E-06	6.125E-06	2.536E-03	2.110E-03
2.812E-01	1.370E-01	6.105E-06	6.106E-06	2.140E-03	2.252E-03
3.125E-01	1.401E-01	6.089E-06	6.089E-06	1.788E-03	2.419E-03
3.438E-01	1.437E-01	6.075E-06	6.076E-06	1.481E-03	2.617E-03
3.750E-01	1.476E-01	6.064E-06	6.065E-06	1.215E-03	2.850E-03
4.062E-01	1.519E-01	6.055E-06	6.056E-06	9.891E-04	3.125E-03
4.375E-01	1.567E-01	6.046E-06	6.046E-06	7.965E-04	3.453E-03
4.688E-01	1.621E-01	6.036E-06	6.036E-06	6.353E-04	3.844E-03
5.000E-01	1.680E-01	6.024E-06	6.024E-06	5.006E-04	4.314E-03

TABLE V. $N = 0.0$, $\kappa = 0.1$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.316E-01	5.411E-02	5.997E-02	2.794E-02	1.531E-01
3.125E-02	1.307E-01	5.386E-02	5.968E-02	2.742E-02	1.533E-01
6.250E-02	1.296E-01	5.371E-02	5.953E-02	2.716E-02	1.519E-01
9.375E-02	1.286E-01	5.317E-02	5.892E-02	2.553E-02	1.528E-01
1.250E-01	1.274E-01	5.263E-02	5.829E-02	2.375E-02	1.543E-01
1.562E-01	1.268E-01	5.215E-02	5.774E-02	2.145E-02	1.579E-01
1.875E-01	1.270E-01	5.177E-02	5.728E-02	1.912E-02	1.634E-01
2.188E-01	1.280E-01	5.153E-02	5.697E-02	1.687E-02	1.712E-01
2.500E-01	1.291E-01	5.132E-02	5.672E-02	1.463E-02	1.804E-01
2.812E-01	1.306E-01	5.117E-02	5.653E-02	1.250E-02	1.922E-01
3.125E-01	1.325E-01	5.089E-02	5.619E-02	1.056E-02	2.060E-01
3.438E-01	1.351E-01	5.056E-02	5.578E-02	8.922E-03	2.218E-01
3.750E-01	1.375E-01	5.012E-02	5.526E-02	7.359E-03	2.415E-01
4.062E-01	1.405E-01	4.981E-02	5.489E-02	5.996E-03	2.653E-01
4.375E-01	1.439E-01	4.991E-02	5.500E-02	4.883E-03	2.929E-01
4.688E-01	1.435E-01	5.020E-02	5.543E-02	3.631E-03	3.290E-01
5.000E-01	1.556E-01	5.070E-02	5.584E-02	3.180E-03	3.713E-01

TABLE VI. $N = 0.5$, $\kappa = 0.1$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.438E-01	7.006E-02	7.664E-02	1.631E-02	2.457E-01
3.125E-02	1.436E-01	6.998E-02	7.655E-02	1.608E-02	2.464E-01
6.250E-02	1.432E-01	6.975E-02	7.629E-02	1.547E-02	2.486E-01
9.375E-02	1.431E-01	6.943E-02	7.593E-02	1.443E-02	2.533E-01
1.250E-01	1.432E-01	6.912E-02	7.557E-02	1.318E-02	2.602E-01
1.562E-01	1.437E-01	6.882E-02	7.523E-02	1.180E-02	2.699E-01
1.875E-01	1.448E-01	6.859E-02	7.495E-02	1.040E-02	2.826E-01
2.188E-01	1.462E-01	6.842E-02	7.475E-02	9.029E-03	2.983E-01
2.500E-01	1.481E-01	6.831E-02	7.461E-02	7.750E-03	3.174E-01
2.812E-01	1.505E-01	6.819E-02	7.446E-02	6.574E-03	3.404E-01
3.125E-01	1.533E-01	6.805E-02	7.427E-02	5.518E-03	3.679E-01
3.438E-01	1.564E-01	6.791E-02	7.411E-02	4.582E-03	4.003E-01
3.750E-01	1.598E-01	6.783E-02	7.399E-02	3.760E-03	4.386E-01
4.062E-01	1.640E-01	6.784E-02	7.399E-02	3.068E-03	4.840E-01
4.375E-01	1.689E-01	6.799E-02	7.414E-02	2.487E-03	5.376E-01
4.688E-01	1.731E-01	6.820E-02	7.439E-02	1.962E-03	6.022E-01
5.000E-01	1.807E-01	6.842E-02	7.459E-02	1.588E-03	6.787E-01

TABLE VII. $N = 1.0$, $\kappa = 0.1$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.612E-01	8.428E-02	9.030E-02	8.716E-03	4.072E-01
3.125E-02	1.631E-01	8.461E-02	9.060E-02	8.861E-03	4.080E-01
6.250E-02	1.618E-01	8.435E-02	9.037E-02	8.276E-03	4.151E-01
9.375E-02	1.624E-01	8.421E-02	9.022E-02	7.712E-03	4.252E-01
1.250E-01	1.634E-01	8.407E-02	9.005E-02	7.022E-03	4.397E-01
1.562E-01	1.649E-01	8.395E-02	8.992E-02	6.269E-03	4.588E-01
1.875E-01	1.668E-01	8.384E-02	8.979E-02	5.503E-03	4.829E-01
2.188E-01	1.691E-01	8.378E-02	8.971E-02	4.763E-03	5.122E-01
2.500E-01	1.719E-01	8.375E-02	8.967E-02	4.072E-03	5.474E-01
2.812E-01	1.751E-01	8.375E-02	8.966E-02	3.443E-03	5.891E-01
3.125E-01	1.788E-01	8.375E-02	8.965E-02	2.879E-03	6.386E-01
3.438E-01	1.830E-01	8.376E-02	8.965E-02	2.385E-03	6.969E-01
3.750E-01	1.876E-01	8.381E-02	8.969E-02	1.957E-03	7.653E-01
4.062E-01	1.929E-01	8.391E-02	8.979E-02	1.593E-03	8.456E-01
4.375E-01	1.986E-01	8.404E-02	8.992E-02	1.284E-03	9.405E-01
4.688E-01	2.048E-01	8.417E-02	9.007E-02	1.023E-03	1.054E+00
5.000E-01	2.119E-01	8.429E-02	9.018E-02	8.073E-04	1.190E+00

TABLE VIII. $N = 0.0$, $\kappa = 0.3$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.638E-01	1.231E-01	1.469E-01	2.870E-02	1.447E-01
3.125E-02	1.629E-01	1.227E-01	1.466E-01	2.825E-02	1.452E-01
6.250E-02	1.620E-01	1.223E-01	1.461E-01	2.773E-02	1.457E-01
9.375E-02	1.613E-01	1.217E-01	1.455E-01	2.604E-02	1.491E-01
1.250E-01	1.600E-01	1.212E-01	1.450E-01	2.406E-02	1.536E-01
1.562E-01	1.598E-01	1.209E-01	1.448E-01	2.170E-02	1.600E-01
1.875E-01	1.603E-01	1.206E-01	1.445E-01	1.919E-02	1.688E-01
2.188E-01	1.609E-01	1.206E-01	1.448E-01	1.674E-02	1.790E-01
2.500E-01	1.627E-01	1.208E-01	1.450E-01	1.441E-02	1.924E-01
2.812E-01	1.651E-01	1.210E-01	1.453E-01	1.240E-02	2.073E-01
3.125E-01	1.673E-01	1.210E-01	1.455E-01	1.040E-02	2.260E-01
3.438E-01	1.697E-01	1.208E-01	1.453E-01	8.631E-03	2.479E-01
3.750E-01	1.725E-01	1.206E-01	1.451E-01	7.139E-03	2.728E-01
4.062E-01	1.755E-01	1.204E-01	1.450E-01	5.749E-03	3.048E-01
4.375E-01	1.778E-01	1.205E-01	1.454E-01	4.562E-03	3.405E-01
4.688E-01	1.801E-01	1.208E-01	1.462E-01	3.489E-03	3.861E-01
5.000E-01	1.907E-01	1.220E-01	1.474E-01	2.910E-03	4.366E-01

TABLE IX. $N = 0.5$, $\kappa = 0.3$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	2.013E-01	1.556E-01	1.762E-01	1.429E-02	2.520E-01
3.125E-02	2.013E-01	1.555E-01	1.762E-01	1.410E-02	2.536E-01
6.250E-02	2.013E-01	1.554E-01	1.761E-01	1.359E-02	2.577E-01
9.375E-02	2.015E-01	1.554E-01	1.762E-01	1.269E-02	2.656E-01
1.250E-01	2.021E-01	1.554E-01	1.765E-01	1.159E-02	2.765E-01
1.562E-01	2.031E-01	1.555E-01	1.768E-01	1.037E-02	2.911E-01
1.875E-01	2.048E-01	1.556E-01	1.773E-01	9.128E-03	3.090E-01
2.188E-01	2.069E-01	1.559E-01	1.778E-01	7.901E-03	3.308E-01
2.500E-01	2.096E-01	1.562E-01	1.784E-01	6.760E-03	3.569E-01
2.812E-01	2.127E-01	1.565E-01	1.790E-01	5.705E-03	3.878E-01
3.125E-01	2.163E-01	1.568E-01	1.796E-01	4.770E-03	4.243E-01
3.438E-01	2.203E-01	1.571E-01	1.801E-01	3.942E-03	4.673E-01
3.750E-01	2.247E-01	1.574E-01	1.808E-01	3.220E-03	5.179E-01
4.062E-01	2.298E-01	1.577E-01	1.814E-01	2.607E-03	5.773E-01
4.375E-01	2.359E-01	1.582E-01	1.822E-01	2.096E-03	6.472E-01
4.688E-01	2.413E-01	1.587E-01	1.830E-01	1.646E-03	7.308E-01
5.000E-01	2.499E-01	1.594E-01	1.839E-01	1.308E-03	8.297E-01

TABLE X. $N = 1.0$, $\kappa = 0.3$

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	2.557E-01	1.882E-01	2.007E-01	6.089E-03	4.906E-01
3.125E-02	2.594E-01	1.881E-01	1.999E-01	6.206E-03	4.901E-01
6.250E-02	2.576E-01	1.882E-01	2.007E-01	5.827E-03	5.021E-01
9.375E-02	2.589E-01	1.884E-01	2.011E-01	5.453E-03	5.174E-01
1.250E-01	2.606E-01	1.886E-01	2.016E-01	4.984E-03	5.395E-01
1.562E-01	2.630E-01	1.889E-01	2.022E-01	4.466E-03	5.681E-01
1.875E-01	2.660E-01	1.893E-01	2.029E-01	3.930E-03	6.036E-01
2.188E-01	2.697E-01	1.897E-01	2.036E-01	3.407E-03	6.457E-01
2.500E-01	2.738E-01	1.902E-01	2.045E-01	2.908E-03	6.975E-01
2.812E-01	2.784E-01	1.906E-01	2.053E-01	2.452E-03	7.583E-01
3.125E-01	2.837E-01	1.911E-01	2.062E-01	2.044E-03	8.292E-01
3.438E-01	2.897E-01	1.916E-01	2.071E-01	1.687E-03	9.117E-01
3.750E-01	2.966E-01	1.922E-01	2.080E-01	1.379E-03	1.008E+00
4.062E-01	3.041E-01	1.928E-01	2.089E-01	1.117E-03	1.120E+00
4.375E-01	3.125E-01	1.934E-01	2.099E-01	8.958E-04	1.252E+00
4.688E-01	3.214E-01	1.940E-01	2.109E-01	7.092E-04	1.408E+00
5.000E-01	3.314E-01	1.947E-01	2.119E-01	5.562E-04	1.590E+00

TABLE XI. $N = 0.0$, $\kappa = 0.5$ (Highly relativistic)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	1.911E-01	1.523E-01	1.833E-01	2.866E-02	1.209E-01
3.125E-02	1.901E-01	1.519E-01	1.831E-01	2.819E-02	1.214E-01
6.250E-02	1.889E-01	1.516E-01	1.829E-01	2.749E-02	1.225E-01
9.375E-02	1.880E-01	1.516E-01	1.836E-01	2.564E-02	1.263E-01
1.250E-01	1.871E-01	1.514E-01	1.837E-01	2.362E-02	1.315E-01
1.562E-01	1.867E-01	1.516E-01	1.846E-01	2.122E-02	1.385E-01
1.875E-01	1.871E-01	1.518E-01	1.854E-01	1.875E-02	1.473E-01
2.188E-01	1.883E-01	1.523E-01	1.865E-01	1.623E-02	1.586E-01
2.500E-01	1.900E-01	1.528E-01	1.877E-01	1.396E-02	1.709E-01
2.812E-01	1.922E-01	1.534E-01	1.890E-01	1.183E-02	1.866E-01
3.125E-01	1.952E-01	1.538E-01	1.898E-01	9.972E-03	2.049E-01
3.438E-01	1.976E-01	1.539E-01	1.903E-01	8.243E-03	2.270E-01
3.750E-01	2.007E-01	1.539E-01	1.908E-01	6.785E-03	2.520E-01
4.062E-01	2.029E-01	1.537E-01	1.912E-01	5.397E-03	2.835E-01
4.375E-01	2.060E-01	1.541E-01	1.922E-01	4.280E-03	3.191E-01
4.688E-01	2.081E-01	1.548E-01	1.940E-01	3.310E-03	3.618E-01
5.000E-01	2.187E-01	1.559E-01	1.951E-01	2.703E-03	4.115E-01

TABLE XII. $N = 0.5$, $\kappa = 0.5$ (Highly relativistic)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	2.589E-01	1.956E-01	2.161E-01	1.217E-02	2.379E-01
3.125E-02	2.586E-01	1.955E-01	2.161E-01	1.199E-02	2.394E-01
6.250E-02	2.586E-01	1.956E-01	2.166E-01	1.154E-02	2.439E-01
9.375E-02	2.590E-01	1.960E-01	2.175E-01	1.079E-02	2.524E-01
1.250E-01	2.598E-01	1.965E-01	2.187E-01	9.869E-03	2.640E-01
1.562E-01	2.613E-01	1.970E-01	2.200E-01	8.839E-03	2.791E-01
1.875E-01	2.633E-01	1.977E-01	2.215E-01	7.779E-03	2.981E-01
2.188E-01	2.660E-01	1.984E-01	2.232E-01	6.733E-03	3.209E-01
2.500E-01	2.694E-01	1.992E-01	2.249E-01	5.750E-03	3.480E-01
2.812E-01	2.728E-01	2.000E-01	2.266E-01	4.839E-03	3.805E-01
3.125E-01	2.769E-01	2.007E-01	2.283E-01	4.029E-03	4.189E-01
3.438E-01	2.814E-01	2.014E-01	2.299E-01	3.313E-03	4.641E-01
3.750E-01	2.861E-01	2.022E-01	2.317E-01	2.688E-03	5.173E-01
4.062E-01	2.920E-01	2.029E-01	2.333E-01	2.164E-03	5.790E-01
4.375E-01	2.990E-01	2.038E-01	2.350E-01	1.730E-03	6.517E-01
4.688E-01	3.051E-01	2.045E-01	2.367E-01	1.350E-03	7.385E-01
5.000E-01	3.141E-01	2.055E-01	2.385E-01	1.058E-03	8.408E-01

TABLE XIII. $N = 1.0$, $\kappa = 0.5$ (Highly relativistic)

\tilde{r}_A	\tilde{J}	\tilde{M}	\tilde{M}_0	$\tilde{\Omega}^2$	f_x
1.000E-08	3.578E-01	2.399E-01	2.437E-01	4.250E-03	5.513E-01
3.125E-02	3.583E-01	2.403E-01	2.440E-01	4.206E-03	5.555E-01
6.250E-02	3.592E-01	2.406E-01	2.446E-01	4.048E-03	5.664E-01
9.375E-02	3.608E-01	2.410E-01	2.455E-01	3.800E-03	5.844E-01
1.250E-01	3.637E-01	2.415E-01	2.465E-01	3.489E-03	6.089E-01
1.562E-01	3.667E-01	2.422E-01	2.479E-01	3.136E-03	6.422E-01
1.875E-01	3.711E-01	2.430E-01	2.494E-01	2.769E-03	6.824E-01
2.188E-01	3.757E-01	2.438E-01	2.511E-01	2.402E-03	7.325E-01
2.500E-01	3.816E-01	2.446E-01	2.528E-01	2.054E-03	7.906E-01
2.812E-01	3.878E-01	2.455E-01	2.546E-01	1.730E-03	8.601E-01
3.125E-01	3.927E-01	2.466E-01	2.569E-01	1.424E-03	9.449E-01
3.438E-01	4.025E-01	2.473E-01	2.582E-01	1.185E-03	1.037E+00
3.750E-01	4.120E-01	2.481E-01	2.598E-01	9.668E-04	1.145E+00
4.062E-01	4.232E-01	2.489E-01	2.613E-01	7.820E-04	1.268E+00
4.375E-01	4.357E-01	2.498E-01	2.628E-01	6.269E-04	1.410E+00
4.688E-01	4.489E-01	2.507E-01	2.644E-01	4.971E-04	1.577E+00
5.000E-01	4.652E-01	2.517E-01	2.659E-01	3.926E-04	1.767E+00





























